

# ON THE K-RING OF THE CLASSIFYING SPACE OF THE GENERALIZED QUATERNION GROUP

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**ABSTRACT.** We describe the K-ring of the classifying space of the generalized quaternion group in terms of generators and the minimal set of relations. We also compute the order of the main generator in the truncated rings.

## 1. INTRODUCTION

The K-ring of the classifying space  $BQ_{2^n}$  of the generalized quaternion group  $Q_{2^n}$ ,  $n \geq 3$ , is described in [2] and [4]. In this note, we will make the description of these rings in a simpler way, by a minimal set of relations on a minimal set of generators. We will also make nice connections of these computations with the computations done for the lens spaces.

In particular, we will compute the order of the main generator in truncated rings, in a much shorter way than that is done in [4].

As usual, the description of these rings will be done from the representation rings of the groups via the Atiyah-Segal Completion Theorem (ASCT) which roughly says that the K-ring of the classifying space of a group is the completion of the representation ring of this group at the augmentation ideal of the representation ring.

We will also check the minimality of the relations we found through the Atiyah-Hirzebruch Spectral Sequence (AHSS), as usual.

In connection with and parallel to this problem, the reader is advised to look at the computations done for cyclic and dihedral groups. A quick survey for the K-Rings of the classifying spaces of cyclic groups, i.e. lens spaces, and of the classifying spaces of dihedral groups can be found in [3].

## 2. REPRESENTATIONS

The quaternion group  $Q_{2^n}$ , where  $n \geq 3$ , is generated by two elements  $x$  and  $y$  with the relations  $x^{2^{n-1}} = 1$ ,  $x^{2^{n-2}} = y^2$  and  $xyx = y$ . Note that  $x$  generates a cyclic group of order  $2^{n-1}$  and  $y$  generates a cyclic group of order 4. Note also that  $x^{2^{n-2}}$ , which is equal to  $y^2$ , generate a cyclic group of order 2. We keep in mind these natural group inclusions of the cyclic groups in  $Q_{2^n}$ .

There are four one dimensional irreducible complex representations of  $Q_{2^n}$ . We will denote them by  $1, \eta_1, \eta_2$  and  $\eta_3$ . They are explained by  $1 \times 1$  matrices in [4].

There are  $2^{n-2} - 1$  two dimensional irreducible complex representations of  $Q_{2^n}$ . We will denote them by  $d_i$  where  $1 \leq i \leq 2^{n-2} - 1$ . Actually,  $d_i$  make sense for any integer  $i$  and this will be clarified below.

Since all we need will be the relations that they can generate, we will not describe these representations by matrices here. The descriptions of these representations by  $2 \times 2$  matrices are given in [4].

Before presenting the relations let us set:  $2^n = 2m = 4k$ . And we have this convention from now on. Note that  $k \geq 2$  and it is a power of 2 too.

Now, we will list all possible relations in the representation ring  $R(Q_{4k})$ . First of all  $\eta_3 = \eta_1\eta_2$ . And since  $\eta_1^2 = \eta_2^2 = 1$ , we also have the relation  $\eta_3^2 = 1$ . For  $d_i$ 's, we have the start  $d_0 = 1 + \eta_1$  and we have the end  $d_k = \eta_2 + \eta_3$ .

The main relation, which is the most important of all, is

$$d_i d_j = d_{i+j} + d_{i-j}.$$

This relation make sense for any integer couple  $i, j$  because of the following fact:  $d_i = d_{m-i}$  for all integers  $i$ .

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And another set of relations are for the products of the one dimensional and two dimensional representations, and they are:  $\eta_1 d_i = d_i$  and  $\eta_2 d_i = d_{k-i}$  for all  $i$ . Since,  $\eta_3 = \eta_1 \eta_2$ , it follows that  $\eta_3 d_i = d_{k-i}$  for all  $i$ , the same as  $\eta_2$ .

We deduce from the relations above that, the representation ring of  $Q_{2^n}$  is just generated by  $\eta_1, \eta_2$  and  $d_1$ , by means of tensor products and direct sums. And the minimal polynomials on  $\eta_1, \eta_2$  and  $d_1$  that define the ring can be found from these relations. But, we will do that in our new variables.

### 3. COHOMOLOGY

Integral cohomology of  $Q_{4k}$ ,  $k \geq 2$ , is the following and can be found in [1]:

$$H^p(BQ_{4k}; Z) = \begin{cases} Z & p = 0 \\ Z_2 \oplus Z_2 & \text{if } p = 4s + 2 \\ Z_{4k} & \text{if } p = 4s, s \geq 1 \\ 0 & \text{if } p \text{ is odd.} \end{cases}$$

Note that the odd dimensional cohomology vanishes. Because of that, the AHSS which converges to  $K(BQ_{4k})$ , collapses on page two so that the  $K$ -ring is completely determined by the integral cohomology and vise versa. Here, we also notice that the cohomology is periodic.

The relations of the cohomology ring can also be found in [1]. Note that these relations are quite different than the relations in the  $K$ -ring. We don't try to find connections between these relations. We will just compare the orders of the elements of  $K(BQ_{4k})$  in the filtrations of the spectral sequence with the sizes of the cohomology groups to prove that the relations are minimal.

### 4. K-RINGS

Corresponding to the representations  $\eta_1, \eta_2$  and  $d_1$ , there are induced vector bundles over the classifying space  $BQ_{4k}$  and we denote them by the same letters. We will set the reduced vector bundles as  $v_1 = \eta_1 - 1$ ,  $v_2 = \eta_2 - 1$  and, most importantly, the main element  $\phi = d_1 - 2$ . Due to the ASCT, the elements  $v_1, v_2$  and  $\phi$  generate  $K(BQ_{4k})$ . All we need is to find the minimal relations on these generators so that the ring is well described.

First off all, since  $\eta_1^2 = \eta_2^2 = 1$ , we have the following relations

$$v_1^2 = -2v \text{ and } v_2^2 = -2v_2 \text{ (Relations 1 \& 2).}$$

We note that the above relations are the standard relations for real line bundles over the classifying spaces. These small and simple relations on  $v_i$ 's explain the cohomology groups  $H^{4s+2}(BQ_{4k}; Z) = Z_2 \oplus Z_2$  which are the  $(4s+2)$ -th filtrations  $E_\infty^{4s+2, -4s-2}$  on the main diagonal of the AHSS. The first  $Z_2$  in the direct sum is generated by  $v_1^s$  and the second  $Z_2$  is generated by  $v_2^s$  where  $s \geq 1$ .

Next we should explain  $Z_{4k}$ 's which occur in the cohomology ring, in other words, we should explain the filtrations  $E_\infty^{4s, -4s}$ ,  $s \geq 1$ , of the AHSS. This will not be easy.

Recall the natural inclusion of  $Z_{2k}$  in  $Q_{4k}$  defined by the element  $x \in Q_{4k}$ . This inclusion rises a natural ring homomorphism  $K(BQ_{4k}) \rightarrow K(BZ_{2k})$ .

Under this homomorphism, the image of the virtual bundle  $d_i - 2$  in  $K(BZ_{2k})$  is  $\eta^i + \eta^{-i} - 2$  where  $\eta$  is the one and only generator of  $K(BZ_{2k})$  and the one and only relation it satisfies is  $\eta^{2k} = 1$ . We set  $w = \eta + \eta^{-1} - 2$  in  $K(BZ_{2k})$ . The element  $w$  generates a subring of  $K(BZ_{2k})$  which is isomorphic to the subring of  $KO(BZ_{2k})$ , solely generated by  $w$ . Actually it is almost isomorphic, except a  $Z_2$  direct summand which is generated by the tautological one dimensional reduced real bundle, traditionally denoted by  $\lambda$ . Note that  $KO(BZ_{2k})$  is the real topological K-theory of the space  $BZ_{2k}$ .

Hence, under the natural homomorphism mentioned above, the image of  $d_i - 2$  in  $K(BZ_{2k})$  is  $\psi^i(w)$  where  $\psi^i$  is the Adams operation of degree  $i$ . Lets recall the effect of (the real) Adams operation of degree  $i$ , on the main generator  $w$  of  $KO(BZ_{2k})$ :

$$\psi^i(w) = \sum_{j=1}^i \frac{\binom{i}{j} \binom{i+j-1}{j}}{\binom{2j-1}{j}} w^j$$

We name the above polynomial "quadratic binomial of degree  $i$ " because of its connection to the real part of a root of unity and because its coefficients are polynomials of  $i^2$ . In particular, under the above ring homomorphism,  $\phi$  maps exactly on  $w$ .

On the other hand, in the ring  $K(BQ_{4k})$ , we have  $d_{k+1} - d_{k-1} = 0$ . Practically, we observe that this gives a polynomial in  $\phi$  of degree  $k+1$ , no one dimensional bundles involved. This is true for any  $d_i$  where  $i$  is odd. They can be written as a polynomial of  $\phi$  and only  $\phi$ .

In the ring  $KO(BZ_{2k})$ , we know that the main relation is  $\psi^{k+1}(w) - \psi^{k-1}(w) = 0$ , [3]. And, we deduce that  $\psi^{k+1}(\phi) - \psi^{k-1}(\phi) = 0$  in the ring  $K(BQ_{4k})$  too. We also conclude that

$$d_i - 2 = \psi^i(\phi) \text{ when } i \text{ is odd.}$$

We can also prove this very important fact from the relations of the representation ring without referring to lens spaces. But, this would take longer. Lens spaces make this tricky.

The polynomials  $g_{2k}(\phi) = \psi^{k+1}(\phi) - \psi^{k-1}(\phi)$  are given by the following series:

$$g_{2k}(\phi) = 4k\phi + \sum_{j=2}^k \frac{2k^2 + j - 1}{(j-1)(2j-1)} \binom{k+j-2}{2j-3} \phi^j + \phi^{k+1}$$

Hence, the following relation is satisfied in  $K(BQ_{4k})$ :

$$g_{2k}(\phi) = 0 \text{ (Relation 3).}$$

From this relation, we deduce that  $\phi$  satisfies a relation in the form

$$4k\phi = f(\phi)\phi^2$$

where  $f(\phi) \in K(BQ_{4k})$  is a virtual bundle generated by  $\phi$ . This explains the fact that the 4-th filtration  $E_\infty^{4,-4}$  on the last page of the main diagonal of AHSS is generated by maybe  $\phi$  (or maybe  $\phi - v_1 - v_2$  etc.) and is isomorphic to  $H^4(BQ_{4k}; \mathbb{Z}) = \mathbb{Z}_{4k}$ . We don't want to speculate much about the spectral sequence.

And by multiplying this relation by powers of  $\phi$ , all  $4s$ -th filtrations on the main diagonal of AHSS, i.e. all groups  $H^{4s}(BQ_{4k}; \mathbb{Z}) = \mathbb{Z}_{4k}$  in the cohomology, are similarly explained.

But, we are still not done! It turns out that the relation 3 is not minimal. We will prove it when we talk about the minimal relation for the cross product  $v_1 v_2$ . We also didn't still explain the products  $v_1 \phi$  and  $v_2 \phi$ . Without these relations, the ring can not be completely described, although more or less the filtrations of the diagonal of the AHSS are explained.

Lets first find the minimal relations for the products  $v_i \phi$  where  $i = 1$  or  $2$ , and explain why they are not needed to occupy any place on the AHSS.

From the relations  $\eta_1 d_1 = d_1$ , it immediately follows that

$$v_1 \phi = -2v_1 \text{ (Relation 4).}$$

This takes care of the product  $v_1 \phi$ . Next we will take care of the product  $v_2 \phi$ . From the relation,  $\eta_2 d_1 = d_{k-1}$ , since  $d_{k-1} = \psi^{k-1}(\phi) + 2$ , we obtain

$$v_2 \phi = \psi^{k-1}(\phi) - \phi - 2v_2 \text{ (Relation 5).}$$

So  $v_i \phi$  where  $i = 1$  or  $2$  are dependent variables, and we don't have to search a place on the AHSS for them.

Finally lets explain what remained, in other words, lets find the minimal relation for the cross product  $v_1 v_2$ . It turns out that the main relation is not the Relation 3, but that one. In fact, we will throw away our favorite relation, the Relation 3, from the minimal set of relations.

We will separate the cases  $n = 3$  and  $n \geq 4$ , since it turns out that  $Q_8$  is a little different than the bigger generalized quaternion groups.

For  $k = 2$ , from the relation  $d_1^2 = d_2 + d_0 = 1 + \eta_1 + \eta_2 + \eta_3$ , we have

$$v_1 v_2 = 4\phi + \phi^2 - 2v_1 - 2v_2 \text{ (Relation 6, for } n = 3\text{).}$$

This is the main relation for  $K(BQ_8)$ . If we multiply this equation by  $\phi + 2$ , an amazing thing happens and we find the Relation 3. In other words, we can kick off the Relation 3 from the minimal list of relations that describes the ring.

For  $n \geq 4$ , starting from the relation  $d_1^2 = d_2 + d_0$ , by using the relation  $d_i^2 = d_{2i} + d_0$  repeatedly, one inside the other, we can obtain the relation  $d_k - d_0 = \psi^k(\phi)$ . And along the way, amusingly, we obtain the polynomials  $\psi^i(\phi)$  where  $i$  is a power of 2, in terms of  $\psi^{\frac{i}{2}}(\phi)$ . And thus, we have

$$v_1 v_2 = \psi^k(\phi) - 2v_2 \text{ (Relation 6, for } n \geq 4\text{).}$$

We will throw away the Relation 3 for  $n \geq 4$  too. We multiply the Relation 6 above by  $\phi + 2$  and then we use the Relations 4 and 5 properly in the equation we obtained, and again amazingly find the Relation 3. This also

removes doubts from the obscure explanations we did above when we derived this relation trickily from the lens spaces.

We sum up everything in:

**Theorem 1:**  $K(BQ_{2^n})$  is generated by  $v_1, v_2$  and  $\phi$  with the minimal set of relations (1),(2),(4),(5) and (6) above.

## 5. ORDERS

In [4], Proposition 5.1, the order of the element  $\phi$  in the truncated ring  $R(Q_{4k})/\phi^2 R(Q_{4k})$ , after a lot of work by huge matrices, is found as  $4k$ . He used these orders, to answer some geometric problems, namely problems about immersion of spaces like  $S^{4N+3}/Q_{4k}$  in real Euclidean spaces. But, from the relation  $4k\phi = f(\phi)\phi^2$  we found above, this is evident.

Similarly, we can find the order of  $\phi$  in the truncated ring  $R(Q_{4k})/\phi^{N+1} R(Q_{4k})$  by a careful counting. In the Relation 3, we observe that the coefficient of  $\phi^2$  is  $\frac{k(2k^2+1)}{3}$  whose primary 2 factor is  $k = 2^{n-2}$ . Since the coefficient of  $\phi$  is  $4k = 2^n$ , the jump between them is  $\frac{2^n}{2^{n-2}} = 4$ . So the total count up to  $\phi^{N+1}$  must be  $4k \cdot 4^N$ . Hence, we have

**Corollary 2:** The order of  $\phi$  in  $K(S^{4N+3}/Q_{4k})$  is  $2^{n+2N}$ .

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